

On the Time Derivative in a Quasilinear Equation

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Dedicated to the memory of *Juha Heinonen* 1960–2007

1 Introduction

The regularity theory for certain parabolic differential equations of the type

$$\frac{\partial u}{\partial t} = \operatorname{div} \mathbf{A}(x, t, u, \nabla u) \quad (1)$$

does often not treat the time derivative u_t , which is regarded as a distribution. Thus the time derivative is a neglected object. In this note we will prove that the weak solutions of the Evolutionary p-Laplace Equation

$$\frac{\partial u}{\partial t} = \operatorname{div}(|\nabla u|^{p-2} \nabla u) \quad (2)$$

have a time derivative u_t in Sobolev's sense. In particular, u_t is not merely a distribution but a measurable function, belonging to some space L^q_{loc} .

No doubt, analogous results are known to the experts. The evident fact is that, if the right-hand side of the equation (the divergence part) is a function, so is the left-hand side (the time derivative). Indeed, it has been noted that this yields a derivative even for systems, as in section 7 of [1], and frequently the required estimates appear at intermediate steps in advanced proofs aiming at the continuity of the gradient ∇u , as in [6]. For equation (2) much simpler proofs are accessible. It is an advantage to have the time derivative at ones disposal at an early stage of the theory. Therefore we have found it worth our while to present a direct and succinct proof of the existence and summability of the time derivative. We are able to avoid the use of Moser's and de Giorgi's iterations. Deeper regularity properties are beyond the scope of this note.

The Evolutionary p-Laplace Equation is *degenerate* for $p > 2$ and *singular* for $1 < p < 2$. We will restrict ourselves to the cases $2 \leq p < \infty$.

We refer to the books [3] and [5] about this equation¹, originally encountered more than half a century ago by Barenblatt. The proof can readily be extended to equations like

$$\frac{\partial u}{\partial t} = \sum_{i,j} \frac{\partial}{\partial x_i} \left(\left| \sum_{k,m} a_{k,m} \frac{\partial u}{\partial x_k} \frac{\partial u}{\partial x_m} \right|^{\frac{p-2}{2}} a_{i,j} \frac{\partial u}{\partial x_j} \right)$$

provided that the constant matrix $(a_{i,j})$ satisfies the ellipticity condition

$$\sum a_{i,j} \xi_i \xi_j \geq \lambda |\xi|^2.$$

Also the case $a_{i,j} = a_{i,j}(t)$ is easy, but further generalizations seem to require more refined assumptions. The result is not valid for all equations of the type (1)². Here we are content with the more pregnant formulation in terms of the Evolutionary p-Laplace Equation.

2 The Caccioppoli Estimate

We first define the concept of solutions, then we state the main theorem. The rest of the section is devoted to a Caccioppoli estimate.

Suppose that Ω is a bounded domain in \mathbf{R}^n and consider the space-time cylinder $\Omega_T = \Omega \times (0, T)$. In the case $p \geq 2$ we say that $u \in L^p(0, T; W^{1,p}(\Omega))$ is a *weak solution* of the Evolutionary p-Laplace Equation, if

$$\int_0^T \int_{\Omega} (-u \phi_t + \langle |\nabla u|^{p-2}, \nabla \phi \rangle) dx dt = 0 \quad (3)$$

for all $\phi \in C_0^1(\Omega_T)$. (The singular case $1 < p < 2$ requires an extra *a priori* assumption, for example, $u \in L^\infty(0, T; L^2(\Omega))$ will do.) In particular, one has

$$\int_0^T \int_{\Omega} (|u|^p + |\nabla u|^p) dx dt < \infty.$$

By the regularity theory one may regard $u(x, t)$ as continuous, a fact which we need not use. The main result is the following.

Theorem 1 *Let $2 \leq p < \infty$. If $u = u(x, t)$ is a weak solution, then the time derivative u_t exists (in Sobolev's sense) and $u_t \in L_{\text{loc}}^{p/(p-1)}(\Omega_T)$.*

¹It is also called the non-Newtonian equation of filtration.

²This may explain why the time derivative is neglected in the literature.

The proof is based on the applicability of the rule

$$\int_0^T \int_{\Omega} u \phi_t dx dt = - \int_0^T \int_{\Omega} \phi \nabla \cdot (|\nabla u|^{p-2} \nabla u) dx dt \quad (4)$$

when $\phi \in C_0^1(\Omega_T)$. Thus the theorem follows provided that it first be properly established that the Sobolev derivatives $\partial/\partial x_j (|\nabla u|^{p-2} \nabla u)$, appearing in the formula, exist and belong to $L_{\text{loc}}^{p/(p-1)}(\Omega_T)$. The main task is thus to prove differentiability in *the x-variable*.

To begin with, we need a variant of the Caccioppoli estimate for the difference $u(x+h, t) - u(x, t)$ where h is a small increment in the desired direction. If ϕ is a given test function with compact support, then also the translated function $v(x, t) = u(x+h, t)$ is a weak solution in some subdomain containing the support of ϕ , provided that $|h|$ is small enough. Subtracting the equations for $u(x, t)$ and $u(x+h, t)$ we obtain

$$\begin{aligned} \int_0^T \int_{\Omega} \langle |\nabla u(x+h, t)|^{p-2} \nabla u(x+h, t) - |\nabla u(x, t)|^{p-2} \nabla u(x, t), \nabla \phi(x, t) \rangle dx dt \\ = \int_0^T \int_{\Omega} (u(x+h, t) - u(x, t)) \phi_t(x, t) dx dt. \end{aligned} \quad (5)$$

Choose the test function

$$\phi(x, t) = \eta(t) \zeta(x)^p (u(x+h, t) - u(x, t))$$

where $\zeta \in C_0^\infty(\Omega)$, $0 \leq \zeta(x) \leq 1$, and $\eta(t)$ is a cut-off function, $0 \leq \eta(t) \leq 1$, and $\eta(0) = \eta(T) = 0$. Strictly speaking it is not an admissible one, because ϕ_t contains the forbidden time derivative u_t . A *formal* calculation yields the *Caccioppoli estimate*³

$$\begin{aligned} \int_0^T \int_{\Omega} \eta(t) \zeta(x)^p \langle |\nabla u(x+h, t)|^{p-2} \nabla u(x+h, t) - |\nabla u(x, t)|^{p-2} \nabla u(x, t), \\ \nabla u(x+h, t) - \nabla u(x, t) \rangle dx dt \\ = -p \int_0^T \int_{\Omega} \eta(t) \zeta(x)^{p-1} (u(x+h, t) - u(x, t)) \\ \times \langle |\nabla u(x+h, t)|^{p-2} \nabla u(x+h, t) - |\nabla u(x, t)|^{p-2} \nabla u(x, t), \nabla \zeta(x, t) \rangle dx dt \\ + \frac{1}{2} \int_0^T \int_{\Omega} \eta'(t) \zeta(x)^p (u(x+h, t) - u(x, t))^2 dx dt \end{aligned}$$

³This is a slight abuse of the name, since there is no estimate yet.

after some integrations by part of the integral containing ϕ_t .

In order to justify the use of the test function above we introduce the convolution

$$f(x, t)^* = \int_0^T \int_{\Omega} f(x - y, t - \tau) \rho_{\sigma}(y, \tau) dy d\tau,$$

where ρ_{σ} is a smooth non-negative function with compact support in the ball $|y|^2 + \tau^2 \leq \sigma^2$; σ is small. (In fact, convolution only in the time variable would suffice. The familiar Steklov average works well.) With the abbreviations $u = u(x, t)$ and $v = u(x + h, t)$ we obtain the averaged identity

$$\begin{aligned} \int_0^T \int_{\Omega} \langle (|\nabla v|^{p-2} \nabla v)^* - (|\nabla u|^{p-2} \nabla u)^*, \nabla \phi \rangle dx dt \\ = \int_0^T \int_{\Omega} (v^* - u^*) \phi_t dx dt \end{aligned} \quad (6)$$

from equation (5). This is a standard procedure. The parameter σ has to be less than a bound depending on $|h|$ and on the distance from the support of the test function ϕ to the boundary. Now we insert the test function

$$\phi(x, t) = \eta(t) \zeta(x)^p (v(x, t)^* - u(x, t)^*)$$

into (6). This is an admissible one. Again the integral containing ϕ_t becomes

$$\frac{1}{2} \int_0^T \int_{\Omega} \eta'(t) \zeta(x)^p (v(x, t)^* - u(x, t)^*)^2 dx dt.$$

Here we may safely let $\sigma \rightarrow 0$. The terms coming from $\nabla \phi$ cause no problem, when $\sigma \rightarrow 0$. Thus we arrive at the Caccioppoli estimate again, but this time the procedure was duly justified.

3 Estimation of Difference Quotients

We aim at proving differentiability in the variable x of the auxiliary vector field

$$F(x, t) = |\nabla u(x, t)|^{(p-2)/2} \nabla u(x, t)$$

by bounding its integrated difference quotients. Notice that we have $(p-2)/2$ in place of the desired exponent $p-2$, the transition to which is explained in section 4. In the stationary case this expedient quantity was employed by Bojarski and Iwaniec, cf. [2]. They used the elementary inequalities

$$\frac{4}{p^2} \left| |b|^{\frac{p-2}{2}} b - |a|^{\frac{p-2}{2}} a \right|^2 \leq \langle |b|^{p-2} b - |a|^{p-2} a, b - a \rangle \quad (7)$$

$$\left| |b|^{p-2}b - |a|^{p-2}a \right| \leq (p-1) \left(|b|^{\frac{p-2}{2}} + |a|^{\frac{p-2}{2}} \right) \left| |b|^{\frac{p-2}{2}}b - |a|^{\frac{p-2}{2}}a \right| \quad (8)$$

for vectors, where $p \geq 2$.⁴

The partial differentiability of F often comes as a by-product of more advanced considerations aiming at establishing the continuity of ∇u itself, as, for example, in [6]. We give a simpler proof below, avoiding iterations. (Needless to say, we do not reach the continuity of ∇u this way.) We write DF for the matrix with the elements

$$\frac{\partial}{\partial x_j} \left(|\nabla u|^{(p-2)/2} \frac{\partial u}{\partial x_i} \right).$$

Lemma 2 *Let $p > 2$. The derivatives DF exist in Sobolev's sense and $DF \in L^2_{loc}(\Omega_T)$. The estimate*

$$\begin{aligned} \int_{\tau}^T \int_{\Omega} \zeta(x)^p |DF|^2 dx dt &\leq \frac{c}{\tau} \int_0^{\tau} \int_{\Omega} \zeta(x)^p |\nabla u(x, t)|^2 dx dt \\ &+ \int_0^T \int_{\Omega} (\zeta(x)^p + |\nabla \zeta(x, t)|^p) |\nabla u(x, t)|^p dx dt \end{aligned} \quad (9)$$

holds when $\tau > 0$. Here $\zeta \in C_0^{\infty}(\Omega)$, $\zeta(x) \geq 0$.

Proof: Proceeding from the Caccioppoli estimate in section 2 we obtain, using the elementary inequalities (7) and (8),

$$\begin{aligned} \frac{4}{p^2} \int_0^T \int_{\Omega} \eta(t) \zeta(x)^p |F(x+h, t) - F(x, t)|^2 dx dt \\ \leq \frac{1}{2} \int_0^T \int_{\Omega} \eta'(t) \zeta(x)^p (u(x+h, t) - u(x, t))^2 dx dt \\ + p(p-1) \int_0^T \int_{\Omega} \left(\eta(t)^{\frac{1}{2}} \zeta(x)^{\frac{p}{2}} |F(x+h, t) - F(x, t)| \right) \\ \times \left(\eta(t)^{\frac{1}{2}} |u(x+h, t) - u(x, t)| |\nabla \zeta(x)| \right) \\ \times \left(|\nabla u(x+h, t)|^{\frac{p-2}{2}} + |\nabla u(x, t)|^{\frac{p-2}{2}} \right) \zeta(x)^{\frac{p-2}{2}} dx dt. \end{aligned} \quad (10)$$

Divide both sides by $|h|^2$ and use the inequality

$$abc \leq \frac{\varepsilon^2 a^2}{2} + \frac{\varepsilon^{-p} b^p}{p} + \frac{(p-2)c^{2p/(p-2)}}{2p}$$

⁴ A proof is worked out in [4].

where the exponents $2, p, 2p/(p-2)$ are conjugated. It follows that the last integral is majorized by

$$\begin{aligned} & \frac{p(p-1)\varepsilon^2}{2} \int_0^T \int_{\Omega} \eta(t) \zeta(x)^p \left| \frac{F(x+h, t) - F(x, t)}{h} \right|^2 dx dt \\ & + (p-1)\varepsilon^{-p} \int_0^T \int_{\Omega} \eta(t)^{\frac{p}{2}} \left| \frac{u(x+h, t) - u(x, t)}{h} \right|^p |\nabla \zeta(x, t)|^p dx dt \\ & + c_p \int_0^T \int_{\Omega} \zeta(x)^p (|\nabla u(x+h, t)|^p + |\nabla u(x, t)|^p) dx dt. \end{aligned}$$

Choose $\varepsilon > 0$ so small that the term with ε^2 is absorbed by the left-hand side of (10), for example, take $p(p-1)\varepsilon^2/2 = 2/p^2$, which is half of $4/p^2$. Then

$$\begin{aligned} & \frac{2}{p^2} \int_0^T \int_{\Omega} \eta(t) \zeta(x)^p \left| \frac{F(x+h, t) - F(x, t)}{h} \right|^2 dx dt \\ & \leq \frac{1}{2} \int_0^T \int_{\Omega} \eta'(t) \zeta(x)^p \left| \frac{u(x+h, t) - u(x, t)}{h} \right|^2 dx dt \\ & + a_p \int_0^T \int_{\Omega} \left| \frac{u(x+h, t) - u(x, t)}{h} \right|^p |\nabla \zeta(x, t)|^p dx dt \\ & + c_p \int_0^T \int_{\Omega} \zeta(x)^p (|\nabla u(x+h, t)|^p + |\nabla u(x, t)|^p) dx dt. \end{aligned}$$

Let us finally select $\eta(t)$ as a piecewise linear cut-off function so that $\eta(t) = 1$ when $\tau \leq t \leq T - \beta$. Since $\eta'(t) < 0$, when $t > T - \beta$, we may omit that portion of the integral in question and then let $\beta \rightarrow 0$. There is no trace left of β in the formula. We can further arrange it so that $\zeta(x) = 1$ in an arbitrary compact subset of Ω . The characterization of Sobolev's space in terms of integrated difference quotients guarantees that the derivatives DF exist. As $h \rightarrow 0$ we arrive at the desired estimate. This concludes the proof.

Remark: Using the 'lost' interval $[T - \beta, T]$ in an effective way, a standard procedure yields an estimate also of

$$\operatorname{ess\,sup}_{0 < t < T} \int_{\Omega} \zeta(x)^p |\nabla u(x, t)|^2 dx.$$

4 The end of the proof in the degenerate case and comments on the singular case

We are in the position to conclude the proof in the case $p > 2$. We have

$$|F|^2 = |\nabla u|^p, \quad |\nabla u|^{p-2} \nabla u = |F|^{1-2/p} F.$$

Thus, in virtue of the lemma,

$$\left| \frac{\partial}{\partial x_j} (|\nabla u|^{p-2} \nabla u) \right| \leq 2 \left(1 - \frac{1}{p} \right) |F|^{\frac{p-2}{p}} \left| \frac{\partial F}{\partial x_j} \right| \quad (11)$$

and, by Hölder's inequality,

$$\frac{\partial}{\partial x_j} (|\nabla u|^{p-2} \nabla u) \in L_{\text{loc}}^{\frac{p}{p-1}}(\Omega_T)$$

because $F \in L^2(\Omega_T)$ and $DF \in L_{\text{loc}}^2(\Omega_T)$. Finally, the theorem follows from the rule (4). This concludes the proof.

Remark: In fact, $F \in L_{\text{loc}}^\infty(\Omega_T)$ and hence one can prove that $u_t \in L_{\text{loc}}^2(\Omega_T)$, which is stronger. However, this boundedness of F requires more advanced regularity theory. For example, in [3] the continuity of ∇u , and consequently of F , is proved.

Let us finally mention that in the *singular case* $1 < p < 2$ one rather easily obtains that the Sobolev derivatives $u_{x_i x_j}$ of the second order and DF exist and belong to $L_{\text{loc}}^2(\Omega_T)$. (When $p > 2$, $u_{x_i x_j}$ is more difficult to achieve!) Unfortunately, one encounters a new complication in (11), caused by the negative exponents. Thus the full regularity theory seems to be needed. In section 2 of [6] the crucial estimate

$$\iint |\nabla u|^{2(p-2)} |D^2 u|^2 dx dt < \infty$$

is given for the range $2 \geq p > \max[3/2, 2n/(n+2)]$. To this one may add that the range $1 < p < 2n/(n+2)$ is not well understood in general.

References

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